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Solution of a singular integral equation

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SUMMARY

A direct method is presented for solving a singular integral equation which is a generalisation of one occurring in viscous flow theory and for which other methods of solution have been described by Brown [1] and Boersma [2].

1. Introduction

The singular integral equation

$$f(x) = \frac{1}{2\pi} \int_0^\infty \log |x-x'| f(x')\mathrm{d}x' + x^{-\frac{1}{2}} \tag{1}$$

was encountered by Van de Vooren and Veldman [3] in a problem in viscous flow theory and an analytic solution subsequently presented by Brown [1]. An alternative method of solution was later described by Boersma [2]. Brown solved the problem using a Wiener-Hopf method which required, in view of the behaviour of $\log |x - x'|$ as $x' \to \infty$, the introduction of a convergence factor $\exp(-\epsilon |x - x'|)$, $(\epsilon > 0)$, into equation (1), whilst Boersma's approach was based on a function-theoretic method first described by Heins and MacCamy [4].

In this paper we describe a direct method of solving a more general singular integral equation, namely,

$$f(x) = \pi^{-1} \int_0^\infty f(x') [\lambda_1 \log |x - x'| + \frac{\lambda_2}{2} \log (x^2 + x'^2 - 2xx' \cos 2\alpha)] \, \mathrm{d}x'$$
$$- \pi^{-1} \lambda_3 x \sin(2\alpha) \int_0^\infty (x^2 + x'^2 - 2xx'\cos 2\alpha)^{-1} f(x') \, \mathrm{d}x' + u(x), \tag{2}$$

where λ_1, λ_2, λ_3 are known constants, $u(x)$ is a known function, $0 < \alpha < \pi$, and $f(x)$ is to be determined. The singular integrals appearing throughout the paper will be understood as their Cauchy Principal Values.

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Equation (2) reduces to equation (1) in either of the two special cases:

(i) $\lambda_1 = \frac{1}{2}$, $\lambda_2 = 0 = \lambda_3$, or, (ii) $\lambda_1 = \lambda_2 = \frac{1}{4}$, $\lambda_3 = 0$, $\alpha = 0$(or π), with $u(x) = x^{-1/2}$.

Equations of the general form of equation (2) arise in the solution of boundary-value problems for Laplace's equation in a wedge-shaped region when a Neumann condition is imposed on one face of the wedge whilst an impedance condition is imposed on the other face.

A Mellin transform is used, in a manner analogous to that adopted by Lighthill [5] to solve an integral equation associated with the jet-flap aerofoil, to transform equation (2) into a difference equation for the Mellin transform of an unknown function ψ, where $f = \mathrm{d}\psi/\mathrm{d}x$. It is shown that this difference equation, which appears at first sight to be rather complicated, can be transformed into a rather simple inhomogeneous difference equation where the inhomogeneous term itself involves the solution of another comparatively simple homogeneous difference equation which we shall call the 'basic' difference equation of the problem. This transformed inhomogeneous difference equation is such that it can be reduced, for an arbitrary inhomogeneous term, to the solution of a standard Carleman-type singular integral equation over a semi-infinite interval. An integral equation of this kind was encountered by Spence [6] in solving an integro-differential equation of the jet-flap aerofoil problem. Spence solved the equation by direct transformation of results for the Carleman equation over a finite interval and though Spence's final result is correct, some of the intermediate analysis gives rise to divergent integrals. We therefore present, in the Appendix, a direct function-theoretic method, avoiding divergent integrals, of solving the singular integral equation of Carleman type over a semi-infinite range.

The final form of solution of the transformed inhomogeneous difference equation is expressed as an integral involving the solution of the 'basic' difference equation and it is shown that this later equation can be transformed into one of a type discussed by Peters [7] (see Stoker [8] also) in solving the 'sloping beach' problem, and a solution is obtained using the method described by Peters.

The solution obtained for general values of α and λ is of a fairly complicated nature mainly due to the rather involved form of the solution of the 'basic' difference equation for the general problem. However, in many particular cases, the solution takes on a simple form and for the function $u(x) = x^{-\frac{1}{2}}$ in equation (2), the solution of equation (1) obtained by previous workers is recovered when the parameters have the values associated with the two particular cases (i) and (ii).

The forms of the solution of the 'basic' difference equation are also explicitly obtained in two further particular cases of equation (2), namely when: (iii) $\lambda_1 = 0$, $\lambda_2 = 1$, $\alpha = \pi/2$, and (iv) $\lambda_1 = \lambda_2 = \lambda_3 = 1$ $(\alpha \neq 0, \pi)$.

2. The method of solution

The general integral equation (2) will be solved by means of Mellin transforms and conditions have to be imposed on f and u as $x \to 0$ and $x \to \infty$, in order to ensure that the various Mellin transforms used exist in a suitable strip.

We shall assume that

$$\begin{aligned} f(x) &= O(x^{-\nu_1}), \quad \text{as} \quad x \to 0 \\ &= o(x^{-(1+\nu_2)}\log x), \quad \text{as} \quad x \to \infty, \end{aligned} \tag{3}$$

and that

$$\begin{aligned} u(x) &= O(x^{1-\nu_3}), \quad \text{as} \quad x \to 0 \\ &= o(x^{-\nu_4}\log x), \quad \text{as} \quad x \to \infty, \end{aligned} \tag{4}$$

with the usual meaning of the 'order' symbols, where $0 < \nu_j < 1$, $(j = 1, 2, 3, 4)$.

The conditions of equation (4) can, by introduction of suitable convergence factors, be relaxed in particular cases; for example the case $u = x^{-\frac{1}{2}}$ can be treated by writing $x^{-\frac{1}{2}}$ as $\lim_{\epsilon \to 0} x^{\frac{1}{2}}/(x + \epsilon)$.

If $\psi(x)$ is such that

$$f(x) = \frac{d\psi}{dx} \equiv \psi'(x), \tag{5}$$

then we note, from equation (3), that ψ can be chosen so that

$$\begin{aligned} \psi(x) &= O(x^{1-\nu_1}), \quad \text{as} \quad x \to 0 \\ &= o(x^{-\nu_2}\log x), \quad \text{as} \quad x \to \infty. \end{aligned} \tag{6}$$

The conditions (4) and (6) then ensure that the Mellin transforms $\Psi(s)$ and $U(s)$, defined by

$$(\Psi(s),\ U(s)) = \int_0^\infty (\psi(x),\ u(x))x^{s-1}dx \tag{7}$$

exist and are analytic in the strip: $\mu - 1 < \mathrm{Re}(s) < \nu$, where $\mu = \mathrm{Max}(\nu_1, \nu_3)$ and $\nu = \mathrm{Min}(\nu_2, \nu_4)$.

Equation (2) becomes, on using equation (5) and integrating by parts,

$$\begin{aligned} \psi'(x) = u(x) &- \frac{\lambda_1}{\pi}\int_0^\infty \frac{\psi(x')dx'}{x' - x} - \frac{\lambda_2}{2\pi}\int_0^\infty \left\{\frac{1}{x' - xe^{i\beta}} + \frac{1}{x' - xe^{-i\beta}}\right\}\psi(x')dx' \\ &- \frac{\lambda_3}{2\pi i}\int_0^\infty \left[\frac{1}{x' - xe^{i\beta}} - \frac{1}{x' - xe^{-i\beta}}\right]\psi'(x')dx', \end{aligned} \tag{8}$$

where $\beta = 2\alpha$.

The Mellin inversion formula gives

$$\psi(x) = +\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Psi(s)x^{-s}ds, \tag{9}$$

where c is assumed to lie in the strip $0 < \mathrm{Re}(s) < \nu < 1$, and therefore, assuming that $s\Psi(s)$ is analytic in the strip $c - 1 < \mathrm{Re}(s) < c$ (this will be verified subsequently),

$$\psi'(x) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} (s-1)\,\Psi(s-1)x^{-s}\mathrm{d}s. \tag{10}$$

Substituting from equations (9) and (10) into equation (8) and using the results

$$\int_0^\infty y^{-s}\mathrm{d}y/(y-x) = \pi\cot(\pi s)x^{-s},$$

$$\int_0^\infty y^{-s}\mathrm{d}y/(y-xe^{i\beta}) = \pi\operatorname{cosec}(\pi s)\,[x\exp\{i(\beta-\pi)\}]^{-s} \tag{11}$$

where $0<\beta<2\pi$, and $0<\mathrm{Re}(s)<1$, shows that equation (8) will be satisfied provided that $\Psi(s)$ satisfies the following functional equation:

$$[\lambda_3\sin(\beta-\pi)s - \sin(\pi s)]\,(s-1)\,\Psi(s-1)$$

$$+\,[\lambda_1\cos(\pi s) + \lambda_2\cos(\beta-\pi)s]\,\Psi(s) = U(s)\sin(\pi s), \tag{12}$$

for s lying in the strip: $0<\mathrm{Re}(s)<\nu$.

Equation (12) simplifies, on making the transformation

$$\Psi(s) = \Gamma(s)\,X(s)\,G(s), \tag{13}$$

where $\Gamma(s)$ is Euler's Gamma function, to the pair:

$$\frac{X(s-1)}{X(s)} = -\tan(\pi s)\frac{\lambda_1\cos(\pi s) + \lambda_2\cos(\beta-\pi)s}{\lambda_3\sin(\beta-\pi)s - \sin(\pi s)} \tag{14}$$

and

$$G(s) - \tan(\pi s)G(s-1) = \frac{U(s)\sin(\pi s)}{\Gamma(s)X(s)\,[\lambda_1\cos(\pi s) + \lambda_2\cos(\beta-\pi)s]}. \tag{15}$$

Equation (14) is the 'basic' difference equation of the problem. The homogeneous form of equation (15) was obtained by Spence and Lighthill in connection with the jet-flap aerofoil problem.

It should be noted that the solution of the 'basic' difference equation (14) is only unique to within a multiplying function of period 1. However, in order to ensure that $s\Psi(s)$ is analytic in the strip $c-1<\mathrm{Re}(s)<c$, by an argument similar to that following equation (35) of Spence's paper, we deduce that this multiplying function must be a constant. If X is multiplied by a constant D then, from equation (15), G will be multiplied by $1/D$ and, since G and X occur in equation (13) as a product, the function Ψ is independent of D and hence there is no loss of generality in assuming D to be unity.

Rewriting the inhomogeneous difference equation (15) as

$$G(s-1) - \cot(\pi s)G(s) = -P(s) \tag{16}$$

and using the convolution theorem for Mellin transforms, viz.

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)H(s)x^{-s}\mathrm{d}s = \int_0^\infty f(y)h(x/y)\,\mathrm{d}y/y, \tag{17}$$

together with the result

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \cot(\pi s)x^{-s}\mathrm{d}s = \frac{1}{\pi(1-x)}, \quad (0<c<\nu<1), \tag{18}$$

gives

$$x^{-1}g(x) - \frac{1}{\pi}\int_0^\infty g(y)\,\mathrm{d}y/(y-x) = -p(x), \quad (x>0) \tag{19}$$

where $g(x)$ and $p(x)$ are the inverse Mellin transforms of the functions $G(s)$ and $P(s)$ respectively, the function $P(s)$ being given by

$$P(s) = U(s)\cos(\pi s)/\Gamma(s)X(s)\{\lambda_1\cos(\pi s) + \lambda_2\cos(\beta-\pi)s\}. \tag{20}$$

The general solution of Carleman equations of the type of equation (19) is obtained in the Appendix.

Using equation (13) and the convolution theorem [cf. equation (17)], we finally obtain the unknown function $\psi(x)$ in the form:

$$\psi(x) = \int_0^\infty g(y)k(x/y)\,\mathrm{d}y/y, \tag{21}$$

where

$$k(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)X(s)x^{-s}\mathrm{d}s. \tag{22}$$

Thus the general solution of equation (2) can be obtained in terms of a solution $X(s)$ of the 'basic' difference equation (14) and the general solution of the Carleman-type equation (19). Whilst the general solution of equation (19) can be obtained from the results presented in the Appendix, it therefore only remains to solve the 'basic' difference equation (15) which simplifies to

$$w(\zeta e^{2\pi i}) - w(\zeta) = m(\zeta), \tag{23}$$

where

$$L(s) = -\tan(\pi s)\cdot\frac{\lambda_1\cos(\pi s) + \lambda_2\cos(\beta-\pi)s}{\lambda_3\sin(\beta-\pi)s - \sin(\pi s)} = \exp[\ell(s)],$$

$$X(s) = \exp[v(s)],$$

$$s = -\frac{1}{2\pi i}\log\zeta, \;\; v(s) = \bar{v}(\zeta), \;\; \ell(s) = \bar{\ell}(\zeta),$$

$$\mu_0 = \lim_{|\zeta|\to\infty}\bar{\ell}(\zeta), \;\; m(\zeta) = \bar{\ell}(\zeta) - \mu_0, \;\; w(\zeta) = \bar{v}(\zeta) - \frac{\mu_0}{2\pi i}\log\zeta. \tag{24}$$

The transformations defined in equation (24) transform the strip $0 < \mathrm{Re}(s) < \nu$ of the complex s-plane into the sector $-2\nu\pi < \arg\zeta < 0$, of the ζ-plane, in which the functional equation (23) is valid. It should be noted at this stage that in order to have μ_0 finite we must require that $\lambda_1 \neq 0$, $\lambda_3 \neq \pm 1$ to include the possibility of α being either zero or π. Assuming the validity of equation (23) in the whole of the cut ζ-plane $-2\nu\pi < \arg\zeta < 2(1-\nu)\pi$, cut along the segment $\arg\zeta = 2(1-\nu)\pi$, and that λ_1 and λ_3 satisfy the above requirements, then a solution can be written down from results due to Peters, as described in Stoker's book [8]. We have that

$$w(\zeta) = \frac{\exp[-2\nu\pi i]}{2\pi i} \int_\infty^0 \frac{m[\xi\exp\{-2\nu\pi i\}]}{\xi\exp\{-2\nu\pi i\} - \zeta}\, d\xi. \tag{25}$$

The function $X(s)$ can now be obtained by the transformations of equation (24), from equation (25). We observe that the form of $w(\zeta)$ in equation (25) ultimately produces a solution $X(s)$ of the 'basic' difference equation (14) which is analytic in the strip $\nu - 1 < \mathrm{Re}(s) < \nu$ of the s-plane. This fact, along with equation (13) and the above-mentioned restrictions on λ_1 and λ_3 verify the assumption on the analyticity of $s\Psi(s)$ in the strip $c - 1 < \mathrm{Re}(s) < c$, with $0 < c < \nu$, made earlier in the analysis.

We now consider some special cases of equation (2) and, in particular, recover the solution obtained by previous workers.

3. The particular cases

If we substitute, in the results obtained above,

case (i) $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \lambda_3 = 0$, $\nu = \frac{1}{2}$, $0 < c < \frac{1}{2}$,

case (ii) $\lambda_3 = 0$, $\lambda_1 = \lambda_2 = \frac{1}{4}$, $\alpha = 0$, or π, $\nu = \frac{1}{2}$,

and take

$$u(x) = x^{\frac{1}{2}}/(x+\epsilon), \quad (\epsilon > 0) \tag{26}$$

and allow $\epsilon \to 0$ after the solution has been obtained, we shall obtain the solution of equation (1) as two particular cases of the general equation (2).

We find that in both the above cases, we have

$$L(s) = \tfrac{1}{2}, \ \mu_0 = -\log 2, \ X(s) = 2^s \tag{27}$$

and

$$k(x) = e^{-x/2}, \ p(x) = (2/\pi x)^{\frac{1}{2}}, \ (\text{when } \epsilon \to 0) \tag{28}$$

so that from equation (5) and equation (21) we obtain

$$f(x) = -\tfrac{1}{2} \int_0^\infty g(1/y) e^{-xy/2}\, dy, \tag{29}$$

the function $g(x)$, satisfying (19), being given by

$$g(x) = -(2/\pi)^{\frac{1}{2}} x^{\frac{1}{2}} + (2/\pi)^{\frac{1}{2}} \frac{x^{3/2} e^{\Omega(x)}}{(1+x^2)^{\frac{1}{2}}}. \tag{30}$$

The relations (29) and (30) finally give the solution of equation (1) in the form

$$f(x) = x^{-\frac{1}{2}} - (2\pi)^{-\frac{1}{2}} \int_0^\infty \frac{e^{\Omega(y)-xy/2}}{y(1+y^2)^{\frac{1}{2}}} \, dy, \tag{31}$$

where

$$\begin{aligned} \Omega(y) &= -\frac{1}{\pi} \int_0^\infty \cot^{-1}(u) \, du/(u-y) \\ &= -\tfrac{1}{2} \log \frac{(1+y^2)^{\frac{1}{2}}}{y} - \frac{1}{\pi} \int_0^y \frac{\log u}{1+u^2} \, du, \end{aligned} \tag{32}$$

and

$$\Omega(1/y) = \Omega(y) - \tfrac{1}{2} \log y.$$

The solution defined by equation (31) of equation (1) is exactly that obtained by Brown and Boersma.

In *case (iii)*, i.e. when $\lambda_1 = 0$, $\lambda_2 = 1$, $\alpha = \pi/2$, the constant λ_1, does not satisfy the requirement necessary for the applicability of Peters' method of solution of the functional equation (14), viz.

$$\frac{X(s-1)}{X(s)} = \frac{1}{\cos(\pi s)}. \tag{33}$$

It is however possible to obtain a solution of equation (33) directly and setting

$$X(s) = \{\cos(\pi s)\}^s \phi(s), \tag{34}$$

shows that ϕ is given by

$$\phi(s) = \exp\{\frac{i\pi s}{2}(s-1)\}. \tag{35}$$

In *case (iv)*, i.e. when $\lambda_1 = \lambda_2 = \lambda_3 = 1$, ($\alpha \neq 0$ or π), the direct approach described through equations (23) to (25) is applicable, and for $\nu = \frac{1}{2}$, we find, after a little manipulation, that we can express $X(s)$ as:

$$X(s) = cX_1(s)/X_0(s) \tag{36}$$

where

$$X_0(s) = [\cos(\pi s)]^{-\frac{1}{2}} \exp\left\{ \int_0^s \frac{(2\pi\theta)\, d\theta}{\sin(2\pi\theta)} \right\} \tag{37}$$

is the Alexeivski function (see Spence [6]), and

$$\log[X_1(s)] = -\frac{1}{4}\log\zeta + \frac{1}{2\pi i}\int_{-\infty}^{0}\frac{\log\left(\frac{\xi^\rho+1}{\xi^\rho-1}\right)d\xi}{\xi-\zeta} \tag{38}$$

with $\arg(\xi) = -\pi$, $\zeta = \exp[-2\pi is]$ and $\rho = (\pi-\alpha)/\pi$, c being an arbitrary constant, which can be chosen to be unity, in order to obtain the solution of (2) by means of the relation (21) [cf. equations (19) and (20) also].

Appendix

The general solution of the singular integral equation

$$c(x)g(x) - \frac{1}{\pi}\int_0^\infty \frac{g(t)\,dt}{t-x} = \ell(x) \tag{I}$$

will be obtained by a direct function-theoretic approach as in Muskhelishvili [9].

We define

$$G(z) = \frac{1}{2\pi i}\int_0^\infty g(\xi)\,d\xi/(\xi-z), \quad z = x+iy. \tag{II}$$

Then, by Plemelj's formulae, (I) reduces to the Hilbert problem

$$[c(x)-i]G^+(x) - [c(x)+i]G^-(x) = \ell(x), \quad (x>0) \tag{III}$$

in the usual notation.

Since

$$\phi(x) = \frac{1}{2\pi i}\log\left[\frac{c(x)+i}{c(x)-i}\right] \to \alpha, \quad (\neq 0, \text{ in general}), \quad \text{as } x\to\infty, \tag{IV}$$

we write the solution of the homogeneous problem (III) in the form:

$$G_0(z) = E(z)z^{-\alpha}\exp[\Omega(z)], \tag{V}$$

where $E(z)$ is an entire function, and

$$\Omega(z) = \int_0^\infty [\phi(\xi)-\alpha]/(\xi-z)\,d\xi, \tag{VI}$$

the function ϕ being given by (IV).

The solution of the inhomogeneous problem (III) is then obtained in the form

$$G(z) = \frac{G_0(z)}{2\pi i}\int_0^\infty \frac{\ell(\xi)/G_0^+(\xi)}{[c(\xi)-i](\xi-z)}\,d\xi + KG_0(z)/z, \tag{VII}$$

where K is an arbitrary constant.

Utilizing the Plemelj formulae once again, we obtain the general solution of (I) in the form:

$$g(x) = \frac{c(x)\ell(x)}{c^2(x)+1} + \frac{e^{\Omega(x)}E(x)}{[c^2(x)+1]^{\frac{1}{2}}} \cdot \frac{1}{\pi}\int_0^\infty \left(\frac{\xi}{x}\right)^\alpha \frac{e^{-\Omega(\xi)}\ell(\xi)\,d\xi}{E(\xi)[c^2(\xi)+1]^{\frac{1}{2}}(\xi-x)}$$

$$+ \frac{K_0 e^{\Omega(x)}E(x)}{x^{\alpha+1}[c^2(x)+1]^{\frac{1}{2}}}, \tag{VIII}$$

K_0 being an arbitrary constant.

In the case of equation (19), we have

$$c(x) = x^{-1},\ \phi(x) = \frac{1}{\pi}\tan^{-1}x - 1,\ (0 < \tan^{-1}x < \tfrac{1}{2}\pi),\ \alpha = -\tfrac{1}{2},$$
$$\Omega(x) = -\frac{1}{\pi}\int_0^\infty \cot^{-1}(u)\,du/(u-x), \tag{IX}$$

and we note that, for equation (1), we have to choose $K_0 = 0$ and $E(z)$ = a constant, in order to satisfy the order conditions (3) and (4), so that with $p(x)$ given by equation (28), we finally obtain the solution of equation (19) in the particular case of equation (1), in the form

$$g(x) = -(2/\pi)^{\frac{1}{2}}\frac{x^{\frac{1}{2}}}{x^2+1} - \frac{x^{3/2}e^{\Omega(x)}}{\pi(1+x^2)^{\frac{1}{2}}}(2/\pi)^{\frac{1}{2}}\int_0^\infty \frac{e^{-\Omega(\xi)}\,d\xi}{(1+\xi^2)^{\frac{1}{2}}(\xi-x)} \tag{X}$$

which simplifies to equation (30), by using the result (see Spence [6])

$$\frac{1}{\pi}\int_0^\infty \frac{e^{-\Omega(\xi)}\,d\xi}{(1+\xi^2)^{\frac{1}{2}}(\xi-x)} = -1 + \frac{xe^{-\Omega(x)}}{(1+x^2)^{\frac{1}{2}}}, \tag{XI}$$

where $\Omega(x)$ is given by (IX). (Note that there is a difference of sign in our $\Omega(x)$ as compared to that of Spence).

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